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Interpolation by Radial Basis Functions on Sobolev Space

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Interpolation by translates of suitable radial basis functions is an important approach towards solving the scattered data problem. However, for a large class of smooth basis functions (including multiquadrics $\phi(x) = (|x|^2 + \lambda^2)^{m-d/2}$, $m > d/2$, $2m - d \notin 2\mathbb{Z}$), the existing theories guarantee the interpolant to approximate well only for a very small class of very smooth approximands. The approximands f need to be extremely smooth. Hence, the purpose of this paper is to study the behavior of interpolation by smooth radial basis functions on larger spaces, especially on the homogeneous Sobolev spaces.

Key Words: radial basis functions; interpolation; Sobolev space; Gaussian; multiquadrics; "shifted" surface splines.

1. INTRODUCTION

We consider approximation of real-valued (underlying) functions f which are known only at a discrete set $X := \{x_1, ..., x_N\}$ in $\mathbb{R}^d$, $d \geqslant 1$. Given data $(x_j, f(x_j))$, $j = 1, ..., N$, the radial basis function approach is to choose a function ϕ: $\mathbb{R}^d \to \mathbb{R}$ and to define an approximant $a_{f,X}$ by

$$a_{f,X}(x) := p(x) + \sum_{j=1}^{N} \alpha_j \phi(x - x_j), \tag{1.1}$$

where $p \in \Pi_m$ and α_j are chosen so that

$$\sum_{j=1}^{N} \alpha_j q(x_j) = 0 \qquad \text{for all} \quad q \in \Pi_m. \tag{1.2}$$

Here Π_m denotes the class of all algebraic polynomials of degree less than m on $\mathbb{R}^d$. In particular, $a_{f,X}$ becomes the so-called ϕ interpolant when $a_{f,X}$ satisfies the conditions

$$a_{f,X}(x_j) = f(x_j), \qquad j = 1, ..., N. \tag{1.3}$$

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For a wide choice of functions ϕ and polynomial orders m, including the case $m=0$, the general conditions on ϕ that ensure the nonsingularity of the system (1.2) and (1.3) have been given by Micchelli [M]. The function ϕ is radial in the sense that $\phi(x)=\Phi(|x|)$, and we assume $\phi=\Phi(|\cdot|)$ to be strictly conditionally positive definite of order m, which implies that the matrix $A:=(\phi(x_i-x_j))_{i,j=1,\ldots,N}$ is positive definite on the subset of vector $u \in \mathbb{R}^N$ satisfying $\sum_{j=1}^N u_j p(x_j)=0$ with $p \in \Pi_m$. For $m>0$, we require X to have the nondegeneracy property for Π_m

$$(p|_X=0,\ p\in\Pi_m) \qquad \text{implies} \quad p\equiv 0. \tag{1.4}$$

For more details, the readers are referred to the papers [MN1, MN2, WS], and the survey papers [D, Bu, and P].

Radial basis function interpolation of scattered data is a frequently used method for multivariate data fitting. The existing studies estimate errors of interpolation for the functions in the space

$$\mathscr{F}_\phi := \left\{ g : |g|_\phi^2 := \int_{\mathbb{R}^d} \frac{|\hat{g}(\theta)|^2}{\hat{\phi}(\gamma)}\, d\theta < \infty \right\} \tag{1.5}$$

which is called "native" function space for ϕ, (see [MN2, WS]). However, for smooth basis functions ϕ (e.g., multiquadrics), these spaces $\mathscr{F}_\phi$ are very small. The approximands need to be extremely smooth. Thus, the purpose of this paper is to explore the approximation behavior of interpolants when the approximands f belong to a larger space, especially to the homogeneous Sobolev space. For $k>0$, the homogeneous Sobolev space is defined by

$$W_p^k(\mathbb{R}^d) := \left\{ f : |f|_{k,p} := \left(\sum_{|\alpha|_1=k} \|D^\alpha f\|_{L_p(\mathbb{R}^d)}^p \right)^{1/p} < \infty \right\}$$

with $1 \leqslant p \leqslant \infty$. In order to discuss the extent to which $a_{f,X}$ approximates f, let us assume that $\Omega \subset \mathbb{R}^d$ is an open bounded domain with cone property over which the error between $a_{f,X}$ and f is measured. For a given set X in $\bar{\Omega}$, we define the "density" of X in $\bar{\Omega}$ to be the number

$$h := h(X;\Omega) := \sup_{x\in\Omega} \min_{x_j \in X} |x-x_j|. \tag{1.6}$$

In fact, in this study, we need stability results on the interpolation process. Therefore, we define the separation distance within X by

$$q := \min_{1\leqslant i\neq j\leqslant N} |x_i-x_j|/2. \tag{1.7}$$

Here and in the sequel, without great loss, we assume that there exists a constant $\rho > 0$ such that

$$h/q \leqslant \rho.$$

This condition asserts that the number of the scattered points in the set X is bounded by ch^{-d}, i.e., $N \leqslant ch^{-d}$, with a constant $c > 0$.

The function ϕ has a generalized Fourier transform in the sense of tempered distribution, and we assume that its Fourier transform $\hat{\phi}$ coincide on $\mathbb{R}^d \backslash 0$ with some continuous function while having a certain type of singularity (necessarily of finite order) at the origin. Hence, $\hat{\phi}$ is of the form

$$|\cdot|^{2m} \hat{\phi} = F > 0, \qquad m > d/2 \qquad \text{and} \qquad F \in L_\infty(\mathbb{R}^d). \tag{1.8}$$

In particular, for a given set X, we adopt the scaled basis functions $\phi_h := \phi(\cdot/h)$ (instead of ϕ) for interpolation, and we use the notation

$$s_{f,X}(x) := p(x) + \sum_{j=1}^{N} \alpha_j \phi_h(x - x_j), \qquad p \in \Pi_m, \tag{1.9}$$

to differentiate from the notation $a_{f,X}$ in (1.1). As a matter of fact, we will see in Section 3 that for certain classes of basis functions ϕ, the interpolant $s_{f,X}$ is identically equal to $a_{f,X}$, under some suitable conditions of ϕ.

Our goal is to prove an approximation power of interpolation $s_{f,X}$ on the homogeneous Sobolev space: Let ϕ be a smooth radial basis function that satisfies the condition (1.8). Then, for every function $f \in W_2^m(\mathbb{R}^d) \cap W_\infty^m(\mathbb{R}^d)$, we have the error bound

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \leqslant ch^{m-d/2}$$

with a constant $c > 0$. Note that $2m$ is the order of singularity of $\hat{\phi}$ at the origin, see (1.8). Furthermore, in Section 3, we apply this estimate to specific radial basis functions. In fact, employing the dilated basis function $\phi(\cdot/h)$ means our analysis is stationary. In this case, the approximation order depends on the order of singularity of $\hat{\phi}$ at the origin. The stationary case was analyzed in great detail in the literature. Among them, the readers are referred to the paper [BR].

The following notations are used throughout this paper. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\theta) := \int_{\mathbb{R}^d} f(t)\, e^{-i\theta \cdot t}\, dt.$$

Also, we use the notation $f^\vee$ for the inverse Fourier transform of a function $f \in L_1(\mathbb{R}^d)$. The Fourier transform can be uniquely extended to the space of tempered distributions on $\mathbb{R}^d$. Several different norms are used. In the case that $\mathbf{g}$ is a matrix or a vector, $\|\mathbf{g}\|_p$, $1 \leqslant p \leqslant \infty$, indicates its p-norm with $1 \leqslant p \leqslant \infty$. For $x \in \mathbb{R}^d$, $|x|$ stands for its Euclidean norm and, for $\alpha \in \mathbb{Z}_+^d := \{\beta \in \mathbb{Z}^d : \beta \geqslant 0\}$, we set

$$\alpha! := \alpha_1! \cdots \alpha_d! \qquad \text{and} \qquad |\alpha|_1 := \sum_{k=1}^{d} \alpha_k.$$

2. INTERPOLATION OF FUNCTIONS IN SOBOLEV SPACE

In this section, we estimate an error bound of $f - s_{f,X}$ with the basis functions ϕ that satisfy the condition in (1.8). If a function f is from the space $W_2^m(\mathbb{R}^d)$, our first step is to find a band-limited part

$$f_H := f_{H,h} := \sigma(h\cdot)^\vee * f \in \mathcal{F}_{\phi_h}, \tag{2.1}$$

where σ: $\mathbb{R}^d \to [0, 1]$ is a nonnegative C^∞-cutoff function whose support σ lies in the Euclidean ball B_1 with $\sigma = 1$ on $B_{1/2}$ and $\|\sigma\|_{L_\infty(\mathbb{R}^d)} = 1$. Then, in error analysis, it is useful to divide $f - s_{f,X}$ into two parts,

$$f - s_{f,X} = (f_H - s_{f_H,X}) + (f_T - s_{f_T,X}), \tag{2.2}$$

where

$$f_T := f_{T,h} := f - \sigma(h\cdot)^\vee * f.$$

From the papers (see, e.g., [WS, MN2]), we cite

LEMMA 2.1. *Let $a_{f,X}$ in (1.1) be an interpolant to f on $X = \{x_1, ..., x_N\}$. Given ϕ and m, for all functions f in the native space $\mathcal{F}_\phi$, there is an error bound of the form*

$$|f(x) - a_{f,X}(x)| \leqslant |f|_\phi \, P_{\phi,X}(x),$$

where $P_{\phi,X}(x)$ is the norm of the error functional, i.e.,

$$P_{\phi,X}(x) = \sup_{|f|_\phi \neq 0} \frac{|f(x) - a_{f,X}(x)|}{|f|_\phi}. \tag{2.3}$$

In the following lemma, we estimate the error $f_H - s_{f_H,X}$.

LEMMA 2.2. *Let $s_{f,X}$ be as in (1.9). Then, for every function $f \in W_2^m(\mathbb{R}^d)$, we have*

$$|f_H(x) - s_{f_H,X}(x)| \leqslant c h^{m-d/2} P_{\phi,X/h}(x/h)\, |f|_{m,2}$$

with a constant $c > 0$ depending on σ and $\hat{\phi}$.

Proof. Note that the function $s_{f_H, X}(h\cdot)$ can be considered as an interpolant (employing the shifts of ϕ) to the scaled function $f_H(h\cdot)$ on X/h, i.e.,

$$s_{f_H, X}(h\cdot) = a_{f_H(h\cdot), X/h}$$

with $a_{f,X}$ in (1.1). Then, since $f_H(h\cdot)$ belongs to the native space $\mathscr{F}_\phi$, Lemma 2.1 can be used directly to derive the bound

$$|f_H(h\cdot) - a_{f_H(h\cdot), X/h}|\,(x/h) \leqslant P_{\phi, X/h}(x/h)\,|f_H(h\cdot)|_\phi.$$

From (2.1), we see that $\widehat{f_H(h\cdot)} = h^{-d}\sigma\hat{f}(\cdot/h)$. Invoking the condition $|\cdot|^{2m}\hat{\phi} = F > 0$ with $F \in L_\infty(\mathbb{R}^d)$, it follows from the explicit formula of the norm $|\cdot|_\phi$ in (1.5) that for every function $f \in W_2^m(\mathbb{R}^d)$,

$$\begin{aligned} |f_H(h\cdot)|_\phi^2 &= \int_{\mathbb{R}^d} \frac{|\widehat{f_H(h\cdot)}(\theta)|^2}{\hat{\phi}(\theta)}\,d\theta \\ &= h^{-2d}\int_{\mathbb{R}^d} \left| \frac{|\theta|^{2m}}{F(\theta)}\,\sigma^2(\theta)\,\hat{f}^2(\theta/h)\right| d\theta \\ &\leqslant ch^{2m-d}\,\|\sigma^2/F\|_{L_\infty(\mathbb{R}^d)}\,|f|_{m,2}^2. \end{aligned}$$

This satisfies the required result because σ is supported on the ball B_1. ∎

We now want to find an error bound for $f_T - s_{f_T, X}$. To do this, we introduce some useful lemmas.

Lemma 2.3. *For every $f \in W_\infty^k(\mathbb{R}^d)$ with $k > 0$, there exists a function $\tilde{f}$ defined by*

$$\tilde{f} := h^{-k} f_T = h^{-k}(f - \sigma(h\cdot)^\vee * f)$$

such that

$$\|\tilde{f}\|_{L_\infty(\mathbb{R}^d)} \leqslant c\,|f|_{k,\infty}$$

with a constant $c > 0$ depending on k.

Proof. Using the identity $\int_{\mathbb{R}^d} \sigma(h\cdot)^\vee(\theta)\,d\theta = \sigma(h\cdot)(0) = 1$ for any $h > 0$, we get

$$(f - \sigma(h\cdot)^\vee * f)(t) = \int_{\mathbb{R}^d} \sigma(h\cdot)^\vee(\theta)(f(t) - f(t-\theta))\,d\theta.$$

By taking Taylor expansion of $f(t-\theta)$ about t, it is obvious that

$$f(t) - f(t-\theta) = -\sum_{0<|\nu|_1<k} (-\theta)^\nu D^\nu f(t)/\nu! - R_k f(t,\theta) \tag{2.4}$$

with the remainder in the integral form

$$R_k f(t,\theta) = \sum_{|\nu|_1 = k} (-\theta)^\nu \int_0^1 k(1-y)^{k-1} D^\nu f(t - y\theta)\, dy/\nu!.$$

Note that since $\int_{\mathbb{R}^d} \theta^\nu \sigma(h\cdot)^\vee(\theta)\, d\theta = (ih)^{|\nu|_1} D^\nu \sigma(0) = 0$ for $\nu \neq 0$, the integral of the first term in (2.4) multiplied by $\sigma(h\cdot)^\vee(\theta)$ is identically zero. Thus, defining

$$\tilde{f}(t) := -h^{-k} \int_{\mathbb{R}^d} \sigma(h\cdot)^\vee(\theta)\, R_k f(t,\theta)\, d\theta,$$

we derive the bound

$$\begin{aligned}
\|\tilde{f}\|_{L_\infty(\mathbb{R}^d)} &\leqslant h^{-k} \int_{\mathbb{R}^d} \|R_k f(\cdot,\theta)\|_{L_\infty(\mathbb{R}^d)}\, |\sigma(h\cdot)^\vee(\theta)|\, d\theta \\
&\leqslant c h^{-k} |f|_{k,\infty} \sum_{|\nu|_1 = k} \int_{\mathbb{R}^d} |\theta^\nu \sigma(h\cdot)^\vee(\theta)|\, d\theta \\
&\leqslant c' |f|_{k,\infty} \sum_{|\nu|_1 = k} \|(D^\nu \sigma)^\vee\|_{L_1(\mathbb{R}^d)}.
\end{aligned}$$

Hence this lemma is proved. ∎

Remark. As a matter of fact, for a given $f \in W_\infty^k(\mathbb{R}^d)$, one may derive the inequality $|\tilde{f}|_{j,\infty} \leqslant c h^{-j} |f|_{k,\infty}$ for $j = 1, ..., k$ with $\tilde{f} = h^{-k}(f - \sigma(h\cdot)^\vee * f)$. However, the estimate in the above lemma is enough for the following analysis.

To find an estimate of $f_T - s_{f_T, X}$, we will use the following arguments: Let $g_{f,X}$ be the interpolant to f on X using the (scaled) Gaussian function

$$\varphi_q(x) := \varphi(x/q) := e^{-|x|^2/q^2} \tag{2.5}$$

with q the separation distance within X as defined in (1.7). In this case $m = 0$ (see (1.2)), and $f_{f,X}$ is of the form

$$g_{f,X}(x) = \sum_{j=1}^N \beta_j \varphi_q(x - x_j). \tag{2.6}$$

It is well known from the literature that the matrix

$$\mathbf{A}_\mathbf{g} := (\phi_q(x_i - x_j))_{i,j=1,...,N}$$

is nonsingular (e.g., see [P]). Then the coefficients $\mathbf{b}_{\mathbf{f}} := (\beta_1, ..., \beta_N)^T$ can be written as

$$\mathbf{b}_{\mathbf{f}} = \mathbf{A}_{\mathbf{g}}^{-1}\mathbf{f},$$

where $\mathbf{f} := (f(x_1), ..., f(x_N))^T$.

PROPOSITION 2.4. *Let X be a q-separated set and the matrix $\mathbf{A}_{\mathbf{g}}$ be defined as above. Then, we have the following properties:*

(a) $\|\mathbf{A}_{\mathbf{g}}^{-1}\|_2 \leqslant c_1$ *for some* $c_1 > 0$,
(b) $\|\mathbf{A}_{\mathbf{g}}^{-1}\|_1 = \|\mathbf{A}_{\mathbf{g}}^{-1}\|_\infty \leqslant c_2 \, \|\mathbf{A}_{\mathbf{g}}^{-1}\|_2$ *for some* $c_2 > 0$.

Proof. The work of Schaback (see Theorem 3.1 in [S1]) shows that there exist positive constants c and $M > 0$ such that

$$\|\mathbf{A}_{\mathbf{g}}^{-1}\|_2 \leqslant c q^d \hat{\varphi}_q^{-1}(M/q).$$

We know that the Fourier transform of the Gaussian φ is of the form $\hat{\varphi} = \pi^{d/2} e^{-|\cdot|^2/4}$, and hence, $\hat{\varphi}_q^{-1} = q^{-d}\hat{\varphi}^{-1}(q\cdot) = q^{-d}\pi^{-d/2}e^{|q\cdot|^2/4}$. This leads to the bound $q^d \hat{\varphi}_q^{-1}(M/q) \leqslant \pi^{-d/2} e^{M^2/4}$, which establishes the proof of (a). Also, the inequality in (b) is proved by a direct application of Theorem 3.11 in the paper [BSW]. The identity $\|\mathbf{A}_{\mathbf{g}}^{-1}\|_1 = \|\mathbf{A}_{\mathbf{g}}^{-1}\|_\infty$ is an obvious consequence of symmetry. ∎

LEMMA 2.5. *Let φ_q be the scaled Gaussian function in* (2.5) *and let $g_{f,X}$ be the interpolant to f on X defined as in* (2.6). *Then, if $f \in L_\infty(\mathbb{R}^d)$, there exist positive constants c_1 and c_2 such that*

$$\|\mathbf{b}_{\mathbf{f}}\|_\infty \leqslant c_1 \, \|\mathbf{A}_{\mathbf{g}}^{-1}\|_2 \, \|f\|_{L_\infty(\mathbb{R}^d)} \leqslant c_2 \, \|f\|_{L_\infty(\mathbb{R}^d)}.$$

Proof. Letting $\mathbf{A}_{\mathbf{g}}^{-1} =: (a_{ij})_{1 \leqslant i, j \leqslant N}$, we can write

$$\mathbf{b}_{\mathbf{f}} = \mathbf{A}_{\mathbf{g}}^{-1}\mathbf{f} = \left(\sum_{j=1}^{N} \alpha_{1j} f_j, ..., \sum_{j=1}^{N} a_{N_j} f_j \right)^T.$$

It follows that

$$\begin{aligned}
\|\mathbf{b}_{\mathbf{f}}\|_\infty &\leqslant \max_{1 \leqslant i \leqslant N} \sum_{j=1}^{N} |a_{ij} f_j| \\
&\leqslant \|f\|_{L_\infty(\mathbb{R}^d)} \max_{1 \leqslant i \leqslant N} \sum_{j=1}^{N} |a_{ij}| \\
&= \|\mathbf{A}_{\mathbf{g}}^{-1}\|_1 \, \|f\|_{L_\infty(\mathbb{R}^d)} \leqslant c \, \|\mathbf{A}_{\mathbf{g}}^{-1}\|_2 \, \|f\|_{L_\infty(\mathbb{R}^d)}
\end{aligned}$$

for some $c > 0$. Since, by Proposition 2.4, the matrix norm $\|\mathbf{A}_{\mathbf{g}}^{-1}\|_2$ is bounded, we get the lemma's claim. ∎

We are now ready to estimate the error $f_T - s_{f_T, X}$.

LEMMA 2.6. *Let $s_{f,X}$ be as in (1.9). Let φ be the Gaussian function in (2.5), and assume that $\hat{\phi}$ decays slower than $\hat{\varphi}$ around ∞ such that $\hat{\varphi}_q / \hat{\phi}_h$ is uniformly bounded. Then, for every $f \in W^m_\infty(\mathbb{R}^d)$, we have*

$$|f_T(x) - s_{f_T, X}(x)| \leqslant c h^{m-d/2} (1 + P_{\phi, X/h}(x/h)) \, |f|_{m,\infty}.$$

Proof. For a given function $f \in W^m_\infty(\mathbb{R}^d)$, by Lemma 2.3, there exist a bounded function $\tilde{f} = h^{-m} f_T$. Then, using the Gaussian interpolant $g_{\tilde{f}, X}$, we have the identity

$$h^{-m}(f_T - s_{f_T, X}) = \tilde{f} - g_{\tilde{f}, X} + (g_{\tilde{f}, X} - s_{\tilde{f}, X})$$

with $g_{\tilde{f}, X}$ in (2.6). Then it is useful to estimate separately those three terms on the right-hand side of the above equation. In fact, since $\|\tilde{f}\|_{L_\infty(\mathbb{R}^d)} \leqslant c\,|f|_{m,\infty}$ by Lemma 2.3, we only need to find bounds of the terms $g_{\tilde{f}, X}$ and $g_{\tilde{f}, X} - s_{\tilde{f}, X}$.

First, in order to evaluate $g_{\tilde{f}, X}$, we apply Lemma 2.3 and Lemma 2.5 to obtain

$$|g_{\tilde{f}, X}(x)| \leqslant \|\mathbf{b}_{\tilde{\mathbf{f}}}\|_\infty \sum_{j=1}^{N} \varphi_q(x - x_j) \leqslant c\,|f|_{m,\infty} \sum_{j=1}^{N} \varphi_q(x - x_j).$$

Here, since X is a q-separated set and the function φ_q decays exponentially, we can easily check that $\sum_{j=1}^{N} \varphi_q(x - x_j)$ is uniformly bounded on Ω. Next, to estimate the term $g_{\tilde{f}, X} - s_{\tilde{f}, X}$, we make use of the interpolation property $\tilde{f}(x_j) = g_{\tilde{f}, X}(x_j)$ for any $j = 1, \ldots, N$ to get the identity

$$s_{\tilde{f}, X} = s_{g_{\tilde{f}, X}, X}.$$

Then, applying the same technique as in the proof of Lemma 2.2 gives the bound

$$|g_{\tilde{f}, X}(x) - s_{g_{\tilde{f}, X}, X}(x)| \leqslant P_{\phi, X/h}(x/h) \, |g_{\tilde{f}, X}(h \cdot)|_\phi.$$

Recalling the explicit form of $|\cdot|_\phi$ in (1.5), we deduce by change of variables that

$$|g_{\tilde{f}, X}(h \cdot)|_\phi^2 = |g_{\tilde{f}, X}|_{\phi_h}^2 \leqslant c \int_{\mathbb{R}^d} \left| \sum_{j=1}^{N} \beta_j e^{i x_j \cdot \theta} \right|^2 \hat{\varphi}_q(\theta) \, d\theta$$

for some $c > 0$, where the inequality is implied by the condition that $\hat{\varphi}_q / \hat{\phi}_h$ is uniformly bounded. Now, we claim that

$$\hat{\varphi}^2_{q/\sqrt{2}} = q^d (\pi/4)^{d/2} = \hat{\varphi}_q.$$

In fact, remembering the Fourier transform $\hat{\varphi}(\theta) = \pi^{d/2} e^{-|\theta|^2/4}$ with φ in (2.5), this is proved by the following direct calculations

$$\hat{\varphi}^2_{q/\sqrt{2}}(\theta) = (q/\sqrt{2})^{2d}\, \hat{\varphi}^2(q\theta\sqrt{2}) = (\pi/2)^d\, e^{-q^2|\theta|^2/4} q^{2d} = q^d (\pi/4)^{d/2}\, \hat{\varphi}_q(\theta).$$

Hence, we use this claim to derive the relation

$$\begin{aligned} |g_{\tilde{f}, X}(H \cdot)|^2_\phi &\leqslant c q^{-d} (\pi/4)^{-d/2} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N \beta_j e^{i x_j \cdot \theta} \hat{\varphi}_{q/\sqrt{2}}(\theta) \right|^2 d\theta \\ &= c q^{-d} (\pi/4)^{-d/2} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N \beta_j \varphi_{q/\sqrt{2}}(x - x_j) \right|^2 d\theta. \end{aligned}$$

Denote

$$K := \left\| \sum_{j=1}^N \varphi_{q/\sqrt{2}}(\cdot - x_j) \right\|_{L_\infty(\mathbb{R}^d)}.$$

The boundness $K < \infty$ is clear due to the decaying property of the Gaussian function $\varphi_{q/\sqrt{2}}$ and the fact that X is a q-separated set. Thus, it leads to the inequalities

$$\begin{aligned} |g_{\tilde{f}, X}(h \cdot)|^2_\phi &\leqslant c K q^{-d} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N \beta_j \varphi_{q/\sqrt{2}}(x - x_j) \right| dx \\ &\leqslant c K N \, \|\mathbf{b}_{\tilde{\mathbf{f}}}\|_\infty \, q^{-d} \, \|\varphi_{q/\sqrt{2}}\|_{L_1(\mathbb{R}^d)} \\ &\leqslant c' h^{-d} \, \|\mathbf{b}_{\tilde{\mathbf{f}}}\|_\infty, \end{aligned}$$

where the last inequality is true by the condition $N \leqslant c h^{-d}$. Therefore, as a consequence of Lemma 2.3 and Lemma 2.5, we establish

$$|\tilde{f}(x) - g_{\tilde{f}, X}(x)| \leqslant c h^{-d/2} P_{\phi, X/h}(x/h)\, |f|_{m, \infty},$$

and it completes the proof. ∎

From Lemma 2.2 and Lemma 2.6, we get the following result.

THEOREM 2.7. *Let $s_{f, X}$ be as in (1.9), and let $\hat{\phi}$ satisfy the condition $|\cdot|^{2m}\, \hat{\phi} = F > 0$ with $m > d/2$ and $F \in L_\infty(\mathbb{R}^d)$. Assume that there exists a constant $\rho > 0$ such that $h/q \leqslant \rho$. Let φ be the Gaussian function (2.5), and assume* *that $\hat{\phi}$ decays slower than $\hat{\varphi}$ around ∞ such that $\hat{\varphi}_q/\hat{\phi}_h$ is uniformly bounded. Then, for every $f \in W_2^m(\mathbb{R}^d) \cap W_\infty^m(\mathbb{R}^d)$, we have an error bound of the form*

$$|f(x) - s_{f,X}(x)| \leqslant ch^{m-d/2}(1 + P_{\phi, X/h}(x/h)),$$

where the constant $c > 0$ depends on $|f|_{m,2}$ and $|f|_{m,\infty}$.

Proof. Recalling (2.2), the proof of this theorem is achieved by direct applications of Lemma 2.2 and Lemma 2.6. ▮

Now, we want to show that the function $P_{\phi, X/h}(\cdot/h)$ is uniformly bounded on Ω provided that the basis function ϕ satisfies the condition in (1.8). The proof technique of this paper is similar to that of Wu and Schaback [WS] (actually, they estimated $P_{\phi,X}$), but our method is simpler than [WS].

LEMMA 2.8. *Let the basis function ϕ satisfy the assumption in* (1.8). *Then there exists a constant $c > 0$ such that for any $X \subset \Omega$, we have*

$$P_{\phi,X}(x/h) \leqslant c, \qquad x \in \Omega.$$

Proof. Let us denote $u(x) := (u_1(x), ..., u_N(x))^T$ as a vector in $\mathbb{R}^N$. Then, the so-called power function in (2.3) can be rewritten as

$$P^2_{\phi,X}(x) = \min_{u \in K_m} \int_{\mathbb{R}^d} \hat{\phi}(\theta) \left| e^{ix\cdot\theta} - \sum_{j=1}^N u_j(x)\, e^{ix_j\cdot\theta} \right|^2 d\theta$$

with the set

$$K_m := \left\{ u = (u_1(x), \ldots, u_N(x))^T \in \mathbb{R}^N \,\middle|\, \sum_{j=1}^N u_j(x)\, p(x_j) = p(x) \text{ for all } p \in \Pi_m \right\}, \tag{2.7}$$

see [WS] for the details. Then there is a vector $\bar{u} = (\bar{u}_1, ..., \bar{u}_N)$ in the admissible set K_m which satisfies the following conditions:

(a) There exists $c_1 > 0$ such that, for any $x \in \Omega$, $\bar{u}_j(x) = 0$ whenever $|x - x_j| > c_1 h$, with h the density of X as in (1.6).

(b) The set $\{(\bar{u}_1(x), ..., \bar{u}_N(x)) \colon x \in \Omega\}$ is bounded in $\ell_1(X)$.

For the examples of such vectors $\bar{u}$, the readers are referred to the paper [L] and [Y2]. Remembering the condition on $\hat{\phi}$ in (1.8), we have

$$P^2_{\phi,X/h}(x/h) \leqslant \|F\|_{L_\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\theta|^{-2m} \left| 1 - \sum_{j=1}^N \bar{u}_j(x)\, e^{i(x_j - x)\cdot\theta/h} \right|^2 d\theta. \tag{2.8}$$

Let $p_{m-1}(x)$ be the Taylor expansion of ex about the origin of degree $m-1$. The polynomial reproducing property of $\bar{u} \in K_m$ in (2.7) implies that

$$\sum_{j=1}^{N} \bar{u}_j(x)[1-p_{m-1}(i(x_j-s)\cdot\theta)]=0.$$

Thus, for $|\theta| \leqslant 1$, by using the properties (a) and (b) of the vector $\bar{u}$, it follows that

$$\begin{aligned} |\theta|^{-m} & \left|1-\sum_{j=1}^{N} \bar{u}_j(x)\, e^{i(x_j-x)\cdot\theta/h}\right| \\ & \leqslant h^{-m}\, |\theta|^{-m} \sum_{j=1}^{N} |\bar{u}_j(x)((x_j-x)\cdot\theta)^m|/m! \\ & \leqslant c \sum_{j=1}^{N} |\bar{u}_j(x)| \leqslant c'. \end{aligned} \tag{2.9}$$

Also, for $|\theta| > 1$, it is immediate that

$$\left|1-\sum_{j=1}^{N} \bar{u}_j(x)\, e^{i(x_j-x)\cdot\theta/h}\right| \leqslant 1+\sum_{j=1}^{N} |\bar{u}_j(x)| \leqslant c. \tag{2.10}$$

Both bounds in (2.9) and (2.10) can be inserted into the expression (2.8) to get

$$P^2_{\phi,X/h}(x/h) \leqslant c\left(1+\int_{|\theta|>1} |\theta|^{-2m}\, d\theta\right) \leqslant c',$$

as a consequence of $2m > d$. Therefore, we obtain the required result. ▮

Remark. In the above lemma, one may have the constant c independent of Ω. However, practically, it depends on the shape of the boundary. Regions with nice and smooth boundaries can allow better constants than irregular boundaries.

We summarize the results of this section:

THEOREM 2.9. *Under the same conditions and notations in Theorem 2.7, we have an error bound of the form*

$$|f(x)-s_{f,X}(x)| \leqslant ch^{m-d/2},$$

where the constant $c > 0$ depends on $|f|_{m,2}$ and $|f|_{m,\infty}$.

3. APPLICATIONS

We now turn to applications to specific radial basis functions. All the examples here are based on the estimates in Lemma 2.8 and Theorem 2.9.

EXAMPLE 3.1. Let the radial basis function ϕ be chosen to be one of the following functions:

(a) $\phi_\lambda := (-1)^{\lceil m-d/2 \rceil} (|x|^2+\lambda^2)^{m-d/2}$, d odd, (multiquadrics),

(b) $\phi_\lambda := (-1)^{m-d/2+1} (|x|^2+\lambda^2)^{m-d/2} \log(|x|^2+\lambda^2)^{1/2}$, d even, ("shifted" surface splines), where $2m > d$ and $\lambda > 0$, and where $\lceil s \rceil$ indicates the smallest integer greater than s. For the purpose of stressing the parameter λ, we use the notation ϕ_λ, and especially we denote $\phi_{\lambda, h}$ as the scaled function, i.e.,

$$\phi_{\lambda, h} := \phi_\lambda(\cdot/h).$$

Then we find that the Fourier transform of ϕ_λ [GS] is of the form

$$\hat{\phi}_\lambda = c_{m,d} \tilde{K}_m(\lambda \cdot) \, |\cdot|^{-2m},$$

where $c_{m,d}$ is a constant depending on m and d, and $\tilde{K}_\nu(|t|) := |t|^\nu K_\nu(|t|) \neq 0$, $t \geqslant 0$, with $K_\nu(|t|)$ the modified Bessel function of order ν [AS]. We note that

$$\tilde{K}_\nu \sim (1 + |\cdot|^{(2\nu-1)/2}) \, e^{-|\cdot|},$$

and it implies that $\hat{\varphi}_q / \hat{\phi}_{\lambda, h}$ is uniformly bounded. Of course, due to Lemma 2.8,

$$P_{\phi_\lambda, X/h}(x/h) \leqslant c, \qquad x \in \Omega.$$

Thus, based on Theorem 2.9, we have the following result.

THEOREM 3.2. *Let ϕ_λ be one of the radial basis functions: multiquadrics and "shifted" surface splines. Then, for every function $f \in W_2^m(\mathbb{R}^d) \cap W_\infty^m(\mathbb{R}^d)$, we have an error bound of the form*

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \leqslant c h^{m-d/2},$$

where the constant $c > 0$ depends on $|f|_{m,2}$ and $|f|_{m,\infty}$.

Remark. As we already discussed, the interpolation scheme in this study is employing the dilated basis function $\phi(\cdot/h)$, which means our analysis is stationary. It is necessary to point out that the approximation order of the stationary case depends on the order of singularity of $\hat{\phi}$ at the origin. Thus, in the case of inverse multiquadric $\phi(x) := (|x|^2 + \lambda^2)^{m-d/2}$ with $0 < m < d/2$, no approximation order is predicted.

Recalling that the interpolant $a_{f,X}$ in (1.1) uses the original (non-scaled) basis function ϕ_λ, we make one observation concerning the interpolants $a_{f,X}$ in relation to $s_{f,X}$.

THEOREM 3.3. *Given a set X, assume that the interpolant $a_{f,X}$ employs the basis function $\phi_{h\lambda}$ instead of ϕ_λ. Then, the interpolant $a_{f,X}$ is identically equal to $s_{f,X}$ which uses $\phi_{\lambda,h}$.*

Proof. It is sufficient to show that the interpolant $s_{f,X}$ using $\phi_{\lambda,h}$ can be represented as a function in the space $\mathrm{span}\{\phi_{h\lambda}(\cdot - x_1), ..., \phi_{h\lambda}(\cdot - x_N)\} + P_m$. Then, by the uniqueness of the solution of the linear system (1.2) and (1.3), this theorem is verified as true. In particular, we treat only the case d is even, because the other case is proved in the exactly same way. For the purpose, we observe that

$$\begin{aligned} s_{f,X}(x) &= \sum_{j=1}^{N} \alpha_j \phi_{\lambda,h}(x - x_j) + p(x) \\ &= h^{-2m+d} \left[\sum_{j=1}^{N} \alpha_j \phi_{h\lambda}(x - x_j) - \log h \sum_{j=1}^{N} \alpha_j (|x - x_j|^2 + (h\lambda)^2)^{m-d/2} \right] \\ &\quad + p(x). \end{aligned}$$

Now, we are going to show that the term $\sum_{j=1}^{N} \alpha_j (|x - x_j|^2 + (h\lambda)^2)^{m-d/2}$ is a polynomial of degree less than m. Note that $m - d/2$ is a positive integer since d is even. Hence, by expanding the term $(|x - x_j|^2 + (h\lambda)^2)^{m-d/2}$, we have the expression

$$\sum_{j=1}^{N} \alpha_j (|x - x_j|^2 + (h\lambda)^2)^{m-d/2} = \sum_{|r+s|_1 \leqslant 2m-d} c_{h,\lambda,r,s} x^r \sum_{j=1}^{N} \alpha_j x_j^s \qquad (3.1)$$

for some suitable constants $c_{h,\lambda,r,s}$ with $r, s \in \mathbb{Z}_+^d$. Due to the condition $\sum_{j=1}^{N} \alpha_j x_j^s = 0$ for $|s|_1 < m$ (see (1.2)), the right hand side of (3.1) is a polynomial of degree $m - d$. ∎

COROLLARY 3.4. *Let ϕ_λ be one of the radial basis functions: multiquadrics and "shifted" surface splines. Let $a_{f,X}$ be the interpolant to f as in* (1.1). *Assume that the parameter λ in ϕ_λ is taken proportional to h. Then, for every $f \in W_2^m(\mathbb{R}^d) \cap W_\infty^m(\mathbb{R}^d)$, we have an error bound of the form*

$$\|f - a_{f,X}\|_{L_\infty(\Omega)} \leqslant c h^{m-d/2}.$$

When $\lambda = 0$ in the above examples (a) and (b), they become the surface splines $\phi := (-1)^{\lceil m-d/2 \rceil} |\cdot|^{2m-d}$ if d is odd, and $\phi := (-1)^{m-d/2+1} |\cdot|^{2m-d} \log |\cdot|$ if d is even, where $2m > d$ in both cases. In this case, $\mathscr{F}_\phi = W_\phi^m(\mathbb{R}^d)$, and hence, we can estimate directly as in Lemma 2.2 without splitting f into two functions f_H and f_T. Then we get the same error bound as in [MN2, WS].

COROLLARY 3.5. *Let ϕ be the surface spline functions. Then, for every $f \in W_2^m(\mathbb{R}^d)$, we have an error bound of the form*

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \leqslant ch^{m-d/2},$$

where c depends on $|f|_{m,2}$.

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